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Semiscalar representations of the Lorentz group and propagation equations

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Abstract. We construct semiscalar linear representations of the inhomogeneous Lorentz group by considering the invariance of linear propagation equations. There is only one semiscalar representation, and the most general linear propagation equation that admits this Lorentz representation is a telegrapher/Maxwell–Cattaneo type equation, whose elementary solutions propagate at the speed of light. Under a Lorentz boost along the x^1 axis, the propagated field variable transforms as $U' = U \exp q[\gamma(v)(vx^1 - c^2t) + c^2t]$. If one imposes $U' = U$, then the Lorentz boost of the propagation equation acquires a velocity-dependent convection-type term. In the Newtonian limit $c \to \infty$, the equation reduces to the Fourier heat equation, and previous results on semiscalar representations of the Galilean group are regained.

1. Introduction

Consider the general linear propagation equation

$$A^{\mu\nu}(x)U_{,\mu\nu} + B^{\mu}(x)U_{,\mu} + D(x)U = 0 \tag{1}$$

where $\mu, \nu, \ldots = 0, 1, \ldots, n$, and $x^\mu = (ct, x^i)$, with $i, j, \ldots = 1, \ldots, n$, and $U_{,\mu} \equiv \partial U/\partial x^\mu$. Special, homogeneous cases of (1) include: the wave equation

$$g^{\mu\nu}U_{,\mu\nu} \equiv -c^{-2}U_{,tt} + \Delta U = 0 \tag{2}$$

where $g_{\mu\nu}$ is the Minkowski metric tensor and Δ is the Laplacian; the Fourier heat equation

$$\Delta U - 2qU_{,t} = 0 \tag{3}$$

the telegrapher (or damped-wave) equation and the Maxwell–Cattaneo heat equation [1], both of which have the form

$$-w^{-2}U_{,tt} + \Delta U - 2qU_{,t} = 0. \tag{4}$$

Wave equation (2) is manifestly Lorentz invariant. The Fourier heat equation (3) turns out to be invariant under the Galilean boost

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t \tag{5}$$

only if the temperature U obeys the transformation law [2, 3]

$$U' = U \exp q[\boldsymbol{x} \cdot \boldsymbol{v} - \tfrac{1}{2}v^2 t] \qquad q = \text{constant}. \tag{6}$$

This multiplier representation of the Galilean group is reminiscent of the phase transformation of the Schrödinger wavefunction under Galilean boost [4].

If, in contrast, one insists on physical grounds that temperature must be an invariant under velocity boosts (see, for example, [5, 6] for discussion of this point), then this result shows that the Fourier heat equation cannot be invariant under Galilean transformations (except in the trivial case $q = 0$). Indeed, invariance of U implies that (3) transforms as follows under a Galilean boost (5):

$$\Delta' U' - 2q' U'_{,t'} + 2q' \boldsymbol{v} \cdot \nabla' U' = 0 \qquad U' = U,\ q' = q. \tag{7}$$

In other words, if temperature is invariant, then the boosted heat equation acquires a velocity-dependent convection term. In fact, we can run the argument backwards, and use the transformation law (6) to remove a convection term from the heat equation (7). This gives a group-theoretic basis for a well known transformation of variables [7].

In this paper we investigate the Lorentz transformation of the linear propagation equation (1) (with the Galilean transformation arising as the limiting case $c \to \infty$). In section 2 we define semiscalar representations as those Lorentz representations for which U is a scalar under translations and rotations, but not necessarily under boosts. We show that there is a unique such proper representation. For a Lorentz boost along the x^1 axis

$$t' = \gamma(v)\left(t - \frac{vx^1}{c^2}\right) \qquad x'^1 = \gamma(v)(x^1 - vt) \qquad \gamma(v) \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{8}$$

we find that

$$U' = U \exp q[\gamma(v)(vx^1 - c^2 t) + c^2 t]. \tag{9}$$

Finally, in section 3 we show that Lorentz invariance reduces (1) to the form

$$-c^{-2} U_{,tt} + \Delta U - 2qU_{,t} + rU = 0 \qquad q, r = \text{constant} \tag{10}$$

and we consider the boost of (10) when $U' = U$ is imposed. The Galilean case of (10) is just the Fourier heat equation (3) (with source term rU), while the Galilean temperature transformation (6) arises from (9) as $c \to \infty$.

2. Semiscalar representations of the Lorentz group

We use the Lie theory [2, 3, 8] and write symmetry generators of equation (1) in the form

$$\mathbf{X} = \Gamma^\mu(x, U)\frac{\partial}{\partial x^\mu} + \Lambda(x, U)\frac{\partial}{\partial U}. \tag{11}$$

The symmetry Lie algebra of any equation (1) contains an infinite-dimensional ideal spanned by

$$\mathbf{X}_\Sigma = \Sigma(x)\frac{\partial}{\partial U} \tag{12}$$

where Σ is an arbitrary solution of the equation (1) in question. In what follows, we will consider the quotient algebra by this ideal.

Our starting point is an important but little known result [3, 8]:

Theorem 1. The general form of symmetry operator (11) for the linear propagation equation (1) is

$$\mathbf{X} = \Gamma^\mu(x)\frac{\partial}{\partial x^\mu} + [\Lambda(x)U + \Sigma(x)]\frac{\partial}{\partial U}$$

where Σ satisfies (1). Hence the quotient algebra by the ideal (12) consists of the symmetry generators of the form

$$\mathbf{X} = \Gamma^{\mu}(x)\frac{\partial}{\partial x^{\mu}} + \Lambda(x)U\frac{\partial}{\partial U}. \tag{13}$$

Consider now the inhomogeneous Lorentz group. It consists of the spacetime translations and space rotations generated by

$$\mathbf{X}_{\mu} = \frac{\partial}{\partial x^{\mu}} \qquad \mathbf{X}_{ij} = x^{j}\frac{\partial}{\partial x^{i}} - x^{i}\frac{\partial}{\partial x^{j}} \tag{14}$$

and the Lorentz boosts generated by

$$\mathbf{Y}_{i} = t\frac{\partial}{\partial x^{i}} + \frac{x^{i}}{c^{2}}\frac{\partial}{\partial t}\,. \tag{15}$$

For the symmetry analysis of (1), Lorentz transformations of spacetime

$$x'^{\mu} = f^{\mu}(x, a) \tag{16}$$

where a is a group parameter, are extended to the propagated field variable U as follows.

$$U' = F(x, a)U. \tag{17}$$

By theorem 1, this is, in fact, the most general allowed form of transformation of U, since it is generated by an operator of the form (13), where

$$\Gamma^{\mu}(x) = \frac{\partial f^{\mu}(x, a)}{\partial a}\bigg|_{a=0} \qquad \Lambda(x) = \frac{\partial F(x, a)}{\partial a}\bigg|_{a=0}.$$

If $F(x, a) = 1$, the variable U is said to be *scalar* under the corresponding transformation (16). Since equation (1) is linear and homogeneous it admits the dilation group with the generator

$$\mathbf{Z} = U\frac{\partial}{\partial U}.$$

We make the following definition (cf the similar definition for the Galilean group in [2]):

Definition. An extension (17) of the Lorentz group to (x^{μ}, U)-space is called a *semiscalar linear representation* if the variable U is a scalar under spacetime translations and space rotations. Thus the infinitesimal generators of a semiscalar representation of the Lorentz group have the form (14) and

$$\mathbf{Y}_{i}^{*} = t\frac{\partial}{\partial x^{i}} + \frac{x^{i}}{c^{2}}\frac{\partial}{\partial t} + \Lambda_{i}(x)U\frac{\partial}{\partial U}.$$

In the particular case $\Lambda_{i}(x) = 0$, the representation is said to be *scalar*.

One can readily carry over the classification of semiscalar representations of the Galilean group [2] to the Lorentz group. The result is as follows.

Theorem 2. There exist two non-similar semiscalar linear representations of the Lorentz group: the scalar representation defined by the generators (14)–(15), and the *proper semiscalar representation* with the generators

$$\mathbf{X}_{\mu} = \frac{\partial}{\partial x^{\mu}} \qquad \mathbf{X}_{ij} = x^{j}\frac{\partial}{\partial x^{i}} - x^{i}\frac{\partial}{\partial x^{j}} \qquad \mathbf{Z} = U\frac{\partial}{\partial U} \tag{18}$$

$$\mathbf{Y}_{i}^{*} = t\frac{\partial}{\partial x^{i}} + \frac{x^{i}}{c^{2}}\frac{\partial}{\partial t} - qx^{i}U\frac{\partial}{\partial U} \qquad q = \text{constant} \neq 0. \tag{19}$$

Note that the constant q is the same for all $\mathbf{Y}_i^*$. Furthermore, one can scale q out of (19) by the dilation $t' = qt,\ x'^i = qx^i$. We can find the transformation of the propagated field variable U under the Lorentz boosts $\mathbf{Y}_i^*$ of (19), e.g. for $i = 1$. The operator $\mathbf{Y}_1^*$ leaves invariant the coordinates $x^2, \ldots, x^n$ while the group transformations (16), (17) of t, x^1, U are determined by the following Lie equations:

$$\begin{aligned}
&\frac{\mathrm{d}t'}{\mathrm{d}a} = \frac{x'^1}{c^2} \qquad && t'|_{a=0} = t \\
&\frac{\mathrm{d}x'^1}{\mathrm{d}a} = t' \qquad && x'^1|_{a=0} = x^1 \\
&\frac{\mathrm{d}U'}{\mathrm{d}a} = -qx'^1 U' \qquad && U'|_{a=0} = U.
\end{aligned}$$

It follows that

$$\begin{aligned}
t' &= t\cosh\left(\frac{a}{c}\right) + \frac{x^1}{c}\sinh\left(\frac{a}{c}\right) \\
x'^1 &= x^1\cosh\left(\frac{a}{c}\right) + ct\sinh\left(\frac{a}{c}\right) \\
U' &= U\exp -q\left[cx^1\sinh\left(\frac{a}{c}\right) + c^2 t\cosh\left(\frac{a}{c}\right) - c^2 t\right].
\end{aligned} \tag{20}$$

By introducing the velocity v of the Lorentz boost:

$$v = -c\tanh\left(\frac{a}{c}\right)$$

one can rewrite the transformations (20) in the form (8), (9):

$$\begin{aligned}
&t' = \gamma(v)\left(t - \frac{vx^1}{c^2}\right) \qquad x' = \gamma(v)(x^1 - vt) \\
&U' = U\exp q[\gamma(v)(vx^1 - c^2 t) + c^2 t].
\end{aligned}$$

3. Lorentz invariant propagation equations

Now we can establish the restrictions on (1) which follow when it is invariant under the Lie algebra spanned by (18), (19):

Theorem 3. The most general linear propagation equation (1) admitting the semiscalar representation of the Lorentz group generated by (18), (19) is (10), i.e.

$$-c^{-2}U_{,tt} + \Delta U - 2qU_{,t} + rU = 0.$$

Proof. Invariance under the generators $\mathbf{X}_\mu$ and $\mathbf{Z}$ of (18) shows that the coefficients $A^{\mu\nu}, B^\mu, D$ of (1) must be constants. Invariance under the rotation generators $\mathbf{X}_{ij}$ then diagonalizes $A^{\mu\nu}$ and forces $A^{ij} = A^{11}\delta^{ij}, B^i = 0$, so that (1) is reduced to the form [2]:

$$A^{00}U_{,tt} + A^{11}\Delta U + B^0 U_{,t} + DU = 0 \qquad A^{00}, A^{11}, B^0, D = \text{constant}. \tag{21}$$

Consider now invariance under $\mathbf{Y}_i^*$. After prolongation [2, 8] to the derivatives involved in (21), the operator has the form:

$$\begin{aligned}
\mathbf{Y}_i^{**} = {}& t\frac{\partial}{\partial x^i} + \frac{x^i}{c^2}\frac{\partial}{\partial t} - qx^i U\frac{\partial}{\partial U} - \left(U_{,i} + qx^i U_{,t}\right)\frac{\partial}{\partial U_{,t}} \\
& -[qx^i U_{,k} + (qU + c^{-2}U_{,t})\delta^i_k]\frac{\partial}{\partial U_{,k}} - (qx^i U_{,tt} + 2U_{,it})\frac{\partial}{\partial U_{,tt}} \\
& -[qx^i U_{,kk} + 2(qU_{,i} + c^{-2}U_{,it})\delta^i_k]\frac{\partial}{\partial U_{,kk}}.
\end{aligned} \tag{22}$$

Invariance of (21) under Lorentz boost is given by the 'infinitesimal' condition:

$$Y_i^{**}(A^{00}U_{,tt} + A^{11}\Delta U + B^0 U_{,t} + DU) = 0 \qquad \text{when (21) holds}$$

and using (22) this yields

$$A^{00} + c^{-2}A^{11} = 0 \qquad B^0 + 2qA^{11} = 0$$

which leads to (10) on re-scaling A^{11} to 1.

It follows that the telegrapher-type equation, with the phase speed of plane-wave solutions equal to the speed of light, is the most general Lorentz invariant propagation equation. Note that the Lorentz invariant equation (10) can only be considered as a Maxwell–Cattaneo heat equation if the phase speed of high-frequency thermal signals is c, which can be the case for radiative heat transfer [9]. For other relativistic heat propagation problems, with sub-luminal thermal signals, (10) is not a physical heat equation.

The propagated field U must transform as (9) under a Lorentz boost. As in the Galilean case discussed in section 1, it follows that if one demands invariance of U on physical grounds, then the equation (10) cannot be invariant under Lorentz transformation (except in the trivial case $q = 0$). In fact, the Lorentz boost of (10) also acquires a convection-type term in the case that U is assumed invariant (scalar representation); i.e. under a Lorentz boost along x^1, (10) becomes

$$-c^{-2}U'_{,t't'} + \Delta' U' - 2q'U'_{,t'} + 2q'vU'_{,1'} + rU' = 0 \qquad U' = U,\ q' = \gamma(v)q. \tag{23}$$

This generalizes the corresponding Galilean boost (7) of the Fourier heat equation—with the additional feature that under a Lorentz boost the diffusion-type coefficient q is no longer an invariant when U is invariant. The convection-type term in (23) may be removed by the transformation (9), thus generalizing the corresponding non-relativistic procedure. Thus a group-theoretic approach gives rise to a method of solving equations of the type (23): any solution of (10) may be transformed via (9) into a solution of (23).

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